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On type II strings in two dimensions

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Abstract: We consider type IIA/B strings in two-dimensions and their projection with respect to the nilpotent space-time supercharge. Based on the ground ring structure, we propose a duality between perturbed type II strings and the topological B-model on deformed Calabi-Yau singularities. Depending on the type II spectra, one has either the conifold or the suspended pinch point geometry. Using the corresponding quiver gauge theory, obtained by D-branes wrapping in the resolved suspended pinch point geometry, we propose the all orders perturbative partition function.

Keywords: Superstrings and Heterotic Strings, Topological Strings.

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1. Introduction

There are by now several examples of equivalence relations between non-critical string theories and closed topological B-model on non-compact Calabi-Yau 3-folds [1]. The basic ingredient in the relation is the string theory ground ring, whose (complexified) defining equations are identified with the Calabi-Yau space. Typically, the ground ring defines a Calabi-Yau singularity, and its deformation corresponds to turning on a string perturbation, such as a cosmological constant or RR flux.

The non-critical $c = 1$ bosonic string at the self-dual radius has been related to the topological B-model on the deformed conifold [2, 3]. Type 0 non-critical $\hat{c} = 1$ strings have been related to the topological B-model on certain Z_2 quotients of the conifold [4] (for further related work see [5 – 7]). The partition functions of the non-critical strings have been matched perturbatively with those of the topological B-models.

By wrapping D-branes on the resolved Calabi-Yau singularities, one obtains a UV description of four-dimensional $\mathcal{N}=1$ quiver gauge theories. Their IR physics is obtained by running duality cascades on the field theory side, or by the geometric transition in the topological picture. The partition function of the non-critical strings is identified with the glueball F-terms of the gauge theory [3].

In this paper we will propose a duality between type II strings in two-dimensions and topological B-model on deformed Calabi-Yau singularities. We will find two types of singularities depending on the choice of the type II spectra. One will be the conifold and the other will be the suspended pinch point singularity. These Calabi-Yau singularities describe the complexified ground ring relations. On the type II string side the deformations of the singularities correspond to turning on RR field background. Using the corresponding quiver gauge theory, obtained by D-branes wrapping in the resolved geometry, we propose the all orders perturbative partition function, which coincides with that of the non-critical $c=1$ bosonic string.

The paper is organized as follows. In section 2 we will study the two-dimensional type IIA and type IIB strings. We will analyze their spectra, ground ring structure and their deformations, and the one-loop partition sums. We will also consider projections of these theories by the space-time supercharge $\mathcal{Q}$, which is a nilpotent BRST-like operator. We will propose a description of perturbed type II strings as a topological B-model, on the (deformed) conifold or suspended pinch point singularities. In section 3 we will discuss the four-dimensional $\mathcal{N}=1$ quiver gauge theory obtained from D-branes wrapping in the suspended pinch point geometry. Using its IR physics, we will suggest that the partition function of the perturbed type II strings and the corresponding topological B-model are given by the partition function of the $c=1$ bosonic string at the self-dual radius.

2. $2d$ type II Strings

The two-dimensional fermionic string is described in the superconformal gauge by a matter superfield X and the super Liouville field Φ. In components they take the form

$$\begin{aligned} X &= x + \theta\psi_x + \bar\theta\bar\psi_x + \theta\bar\theta F_x\,, \\ \Phi &= \phi + \theta\psi_l + \bar\theta\bar\psi_l + \theta\bar\theta F_l\,. \end{aligned} \tag{2.1}$$

The field X is free, while Φ has a background charge $Q=2$. The relevant OPEs are

$$\phi(z)\phi(0) = x(z)x(0) \sim -\log(z)\,, \qquad \psi_{l,x}(z)\psi_{l,x}(0) \sim \frac{1}{z}\,. \tag{2.2}$$

Let us consider the GSO projections of the type IIA/B theories and the respective physical spectra. We will consider x compactified on a circle of radius $R=2$, and $\alpha'=2$, which is the point consistent with $\mathcal{N}=2$ supersymmetry on the worldsheet [8]. The GSO projection is imposed by requiring the physical vertex operators to be local with respect to the space-time supercharges. For the type IIB theory the current will be given by

$$S^+(z) = e^{-\frac{\sigma}{2}+\frac{iH}{2}+iX_L}\,, \tag{2.3}$$

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in the left moving sector, and by

$$\bar{S}^{+}(\bar{z}) = e^{-\frac{\bar{\sigma}}{2}+\frac{i\bar{H}}{2}+iX_R}\,, \tag{2.4}$$

in the right moving sector. For the type IIA theory, we require locality with respect to $S^{+}(z)$ defined by (2.3) in the left moving sector, and with respect to

$$\bar{S}^{-}(\bar{z}) = e^{-\frac{\bar{\sigma}}{2}-\frac{i\bar{H}}{2}-iX_R}\,, \tag{2.5}$$

in the right moving sector.

σ is the bosonized superconformal ghost, $\beta\gamma = \partial\sigma$, and H is obtained by bosonizing the fermions $\psi_x\psi_l = \partial H$. In our conventions the conformal dimensions are as follows

$$\left[e^{ikX}\right] = \frac{k^2}{2}\,, \qquad \left[e^{-l\sigma}\right] = -\frac{l^2}{2}+l, \quad \left[e^{inH}\right] = \frac{n^2}{2}\,, \qquad \left[e^{\alpha\phi}\right] = -\frac{1}{2}\alpha(\alpha-2)\,. \tag{2.6}$$

2.1 Type IIB

The spectrum is determined by requiring locality with respect to the supercharges, level matching and mutual locality. However, in the case at hand this does not define the spectrum uniquely. In addition we have the freedom to choose the lowest lying NSNS state to be a momentum or a winding state. The theory perturbed by these lowest lying states is then either type II on the trumpet geometry, for the momentum perturbation, or type II on the cigar geometry, for the winding perturbation. The latter case was analyzed in [9]. For the sake of completeness this section will cover all the available spectra.

Momentum background. Due to the absence of transverse excitations the spectrum is very simple. In the NS sector the tachyon vertex operators are

$$T_k = e^{-\sigma+ikX+\beta\phi}\,. \tag{2.7}$$

Locality with respect to the space-time supercharge projects onto half-integer values for the momentum in the x-direction in both the left moving and the right moving sector. Thus,

$$X : k_L, k_R \in \mathbb{Z}+\frac{1}{2}\,. \tag{2.8}$$

In order to obtain the closed string states we need to combine the left and right moving states in such a way that the momentum along the Liouville direction, which is non-compact, is the same in both sectors.

The half-integer moding does not allow for winding states in the NSNS sector. This is because locality with respect to the lowest lying NSNS momentum state implies that the left and right moving momenta are related by [10]

$$k_L - k_R \in 2Z\,, \tag{2.9}$$

and we can write $k_{L,R} = m \pm n + 1/2$. Level matching implies that $k_L^2 = k_R^2$, and the only possible solution is $n = 0$.

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In the Ramond sector we have the chiral vertex operators

$$V_k = e^{-\frac{\sigma}{2}+\frac{i}{2}\epsilon H+ikX+\beta\phi}\,, \tag{2.10}$$

and the GSO projection implies $k \in \mathbb{Z}$ for $\epsilon = 1$ and $k \in Z + 1/2$ for $\epsilon = -1$, in both left and right sectors.

Before imposing the Dirac equation constraint we find the spectrum:

$$\begin{array}{rll}
NSNS & e^{-\sigma-\bar\sigma+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\
RR,\,\epsilon_L=\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L+X_R)}\,, & m\in\mathbb{Z}\,,\\
 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L-X_R)}\,, & m\neq 0\,,\\
RR,\,\epsilon_L=\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}-\frac{i}{2}\left(H+\bar H\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\
NSR & e^{-\sigma-\frac{\bar\sigma}{2}-i\frac{\bar H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\
 & e^{-\frac{\sigma}{2}-\bar\sigma-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,
\end{array} \tag{2.11}$$

where we have omitted the Liouville dressing. The Liouville dressing is determined by requiring conformal invariance of the integrated vertex operators. In our conventions this requires the coefficient β to be a solution to the equation

$$\frac{k^2}{2} - \frac{1}{2}\beta\,(\beta-2) = \frac{1}{2}\,. \tag{2.12}$$

Furthermore the locality constraint found in [11] requires $\beta \leq \frac{Q}{2}$. Note, however, that here BRST invariance does not follow from conformal invariance alone [12]. Invariance with respect to the susy BRST operator $Q_{\rm susy} = \oint \gamma T_F$ imposes an additional constraint, equivalent to the space-time Dirac equation. Thus, the physical states satisfy in addition $\beta - \frac{Q}{2} = -\epsilon k$. Combining this with the relation following from conformal invariance we conclude that for physical momenta in the Ramond sector $|k| = \epsilon k$.

The spectrum consistent with all the above constraints is given by

$$\begin{array}{rll}
NSNS & e^{-\sigma-\bar\sigma+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\
RR,\,\epsilon_L=\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L+X_R)}\,, & m\geq 0\,,\\
RR,\,\epsilon_L=\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}-\frac{i}{2}\left(H+\bar H\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n<0\,,\\
NSR & e^{-\sigma-\frac{\bar\sigma}{2}-i\frac{\bar H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\geq 0\,,\\
 & e^{-\frac{\sigma}{2}-\bar\sigma-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n<0\,.
\end{array} \tag{2.13}$$

Winding background. If we choose instead to have winding modes in the NSNS sector we find the following spectrum [9, 13]

$$\begin{array}{rll}
NSNS & e^{-\sigma-\bar\sigma+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\
RR,\,\epsilon_L=\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L+X_R)}\,, & m\in\mathbb{Z}\,,\\
 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L-X_R)}\,, & m\neq 0\,,\\
RR,\,\epsilon_L=\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}-\frac{i}{2}\left(H+\bar H\right)+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\
NSR & e^{-\sigma-\frac{\bar\sigma}{2}-i\frac{\bar H}{2}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\
 & e^{-\frac{\sigma}{2}-\bar\sigma-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,.
\end{array}$$

Again, after imposing the Dirac condition the physical spectrum is given by

$$\begin{array}{rll} NSNS & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\ RR,\, \epsilon_L=\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L+X_R)}\,, & m\geq 0\,,\\ NSR & e^{-\sigma-\frac{\bar{\sigma}}{2}-i\frac{\bar{H}}{2}-i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\geq 0\,,\\ & e^{-\frac{\sigma}{2}-\bar{\sigma}-i\frac{H}{2}-i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\geq 0\,. \end{array}$$

2.2 Type IIA

The type IIA NSNS sector is the same as the type IIB theory one. In the right moving Ramond sector we find a modification: operators with $\epsilon_R = 1$ have momentum $k_R \in \mathbb{Z}+\frac{1}{2}$, whereas operators with $\epsilon_R = -1$ have momentum $k_R \in \mathbb{Z}$.

Momentum background The spectrum is given by

$$\begin{array}{rll} NSNS & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\ RR,\, \epsilon_L=1,\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, & m\in\mathbb{Z}\,,\\ & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L+X_R)}\,, & m\neq 0\,,\\ RR,\, \epsilon_L=-1,\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\ NSR & e^{-\sigma-\frac{\bar{\sigma}}{2}+i\frac{\bar{H}}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\ & e^{-\frac{\sigma}{2}-\bar{\sigma}-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,. \end{array} \tag{2.14}$$

After imposing the Dirac equation we are left with

$$\begin{array}{rll} NSNS & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\ RR,\, \epsilon_L=1,\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, & m\geq 0\,,\\ NSR & e^{-\sigma-\frac{\bar{\sigma}}{2}+i\frac{\bar{H}}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n<0\,,\\ & e^{-\frac{\sigma}{2}-\bar{\sigma}-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n<0\,. \end{array} \tag{2.15}$$

Winding background As in the type IIB case we can also start with winding modes in the NSNS sector. The consistent spectrum is then easily derived to be given by

$$\begin{array}{rll} NSNS & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\ RR,\, \epsilon_L=1,\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, & m\in\mathbb{Z}\,,\\ & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L+X_R)}\,, & m\neq 0\,,\\ RR,\, \epsilon_L=-1,\epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\ NSR & e^{-\sigma-\frac{\bar{\sigma}}{2}+i\frac{\bar{H}}{2}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,,\\ & e^{-\frac{\sigma}{2}-\bar{\sigma}-i\frac{H}{2}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\in\mathbb{Z}\,. \end{array} \tag{2.16}$$

Similarly, the Dirac condition now leads to the following spectrum

$$\begin{array}{rll} NSNS & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\in\mathbb{Z}\,,\\ RR,\, \epsilon_L=1,\epsilon_R=-1 & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, & m\geq 0\,, \end{array}$$

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$$
\begin{array}{llr}
RR,\ \epsilon_L=-1, \epsilon_R=1 & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}-\frac{i}{2}\left(H-\bar H\right)-i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, & n\geq 0\,, \\
\qquad NSR & e^{-\sigma-\frac{\bar\sigma}{2}+i\frac{\bar H}{2}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\geq 0\,, \\
 & e^{-\frac{\sigma}{2}-\bar\sigma-i\frac{H}{2}-i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, & n\geq 0\,.
\end{array}
\tag{2.17}
$$

We note in passing that different conventions for the supercharges which replace $S^+ \leftrightarrow S^-$ would result in a set of equivalent theories.

2.3 Ground ring

The spin zero ghost number zero BRST invariant operators generate a commutative, associative ring

$$
\mathcal{O}(z)\mathcal{O}'(0) \sim \mathcal{O}''(0) + \{Q_B, \dots\}\,, \tag{2.18}
$$

called the ground ring, where Q_B is the $N=1$ BRST operator.

In the following we will analyze the ground ring for the type II. The relevant BRST analysis has been performed in [14–16].

Consider the left sector. The chiral BRST cohomology of dimension zero and ghost number zero is given by the infinite set of states $\Psi_{(r,s)}$ with r, s negative integers

$$
\Psi_{(r,s)} \sim O_{r,s} e^{(ik_{r,s}x_L - p_{r,s}\Phi_L)}\,. \tag{2.19}
$$

The Liouville and matter momentum are given by

$$
k_{r,s} = \frac{1}{2}(r-s), \quad p_{r,s} = \frac{1}{2}(r+s+2)\,. \tag{2.20}
$$

The operators $\Psi_{(r,s)}$ are in the NS-sector if $k_{r,s} = (r-s)/2$ takes integer values, and in the R-sector if it takes half integer values.

Of particular relevance for us are the R-sector operators:

$$
\begin{aligned}
x(z) &\equiv \Psi_{(-1,-2)}(z) = \left(e^{-\frac{i}{2}H}e^{-\frac{1}{2}\sigma} - \frac{1}{\sqrt{2}}e^{\frac{i}{2}H}\partial\xi e^{-\frac{3}{2}\sigma}c\right)e^{\frac{i}{2}x-\frac{1}{2}\phi}\,, \\
y(z) &\equiv \Psi_{(-2,-1)}(z) = \left(e^{\frac{i}{2}H}e^{-\frac{1}{2}\sigma} - \frac{1}{\sqrt{2}}e^{-\frac{i}{2}H}\partial\xi e^{-\frac{3}{2}\sigma}c\right)e^{-\frac{i}{2}x-\frac{1}{2}\phi}\,,
\end{aligned}
\tag{2.21}
$$

and the NS-sector operators

$$
u(z) \equiv \Psi_{(-1,-3)}(z) = x^2\,, \qquad v(z) \equiv \Psi_{(-3,-1)}(z) = y^2\,, \qquad w(z) \equiv \Psi_{(-2,-2)}(z) = xy\,, \tag{2.22}
$$

given by

$$
\begin{aligned}
u(z) &= \left(-e^{-iH}e^{-\sigma} + \frac{i}{\sqrt{2}}c\partial\xi\partial(x-i\phi)e^{-2\sigma} - \sqrt{2}c(\partial^2\xi - \partial\xi\partial\sigma)e^{-2\sigma}\right)e^{ix-\phi}\,, \\
w(z) &= \left(\frac{2\sqrt{2}}{3}i\partial H\, e^{-\sigma} + \frac{i}{3}c\partial\xi\left[\partial(x+i\phi)e^{-iH} - \partial(x-i\phi)e^{iH}\right]e^{-2\sigma} + \right. \\
&\qquad \left. + c(\partial^2\xi - \partial\xi\partial\sigma)(e^{iH}+e^{-iH})e^{-2\sigma} - c\partial\xi\partial(e^{iH}+e^{-iH})e^{-2\sigma}\right)e^{-\phi}\,, \\
v(z) &= \left(-e^{iH}e^{-\sigma} + \frac{i}{\sqrt{2}}c\partial\xi\partial(x+i\phi)e^{-2\sigma} + \sqrt{2}c(\partial^2\xi - \partial\xi\partial\sigma)e^{-2\sigma}\right)e^{-ix-\phi}\,.
\end{aligned}
\tag{2.23}
$$

One has the multiplication rule

$$(\Psi_{(r,s)}\Psi_{(r',s')})(z) \sim \Psi_{(r+r'+1,s+s'+1)}(z)\,, \tag{2.24}$$

where $\sim$ indicates that the right hand side could be multiplied by a vanishing constant. The left sector ring of spin zero, ghost number zero BRST invariant operators is generated by the elements x and y

$$\Psi_{(r,s)} = x^{-s-1}y^{-r-1}\,, \qquad r,s \in \mathbb{Z}_-\,. \tag{2.25}$$

Similarly, one can construct the ring in the right sector. In order to construct the ground ring, we combine the left and right sectors with the same left and right Liouville momenta. We define

$$a_{ij} = \begin{pmatrix} x\bar{x} & x\bar{y} \\ y\bar{x} & y\bar{y} \end{pmatrix}, \qquad b_{ij} = \begin{pmatrix} u\bar{u} & u\bar{w} & u\bar{v} \\ w\bar{u} & w\bar{w} & w\bar{v} \\ v\bar{u} & v\bar{w} & v\bar{v} \end{pmatrix}. \tag{2.26}$$

Next we need to impose the GSO projection.

2.3.1 Type IIB

We project with respect to $S^+(z)$ defined by (2.3) in the left moving sector and with respect to $\bar{S}^+(\bar{z})$ defined by (2.4) in the right moving sector. The ground ring elements that survive the GSO projection are

$$a_{ij} = \begin{pmatrix} \mathbf{x\bar{x}} & \\ & \end{pmatrix}, \qquad b_{kl} = \begin{pmatrix} u\bar{u} & & \mathbf{u\bar{v}} \\ & & \\ \mathbf{v\bar{u}} & & \mathbf{v\bar{v}} \end{pmatrix}, \tag{2.27}$$

where we denoted by bold letters the generators of the ground ring. For instance, b_{11} is generated as $a_{11}^2 = b_{11}$. In addition, we have to check locality with respect to the spectra we found in section 2.1.

Momentum background, IIB$_m$ In this case we find that all the elements surviving the GSO projection are local with respect to the spectrum. Thus, the ground ring is generated by four elements $a_{11}, b_{33}, b_{13}, b_{31}$ with the relation

$$(a_{11})^2 b_{33} - b_{13}b_{31} = 0\,. \tag{2.28}$$

Note that a_{11} is in the RR sector while the b_{ij} are in the NSNS sector. Note also that a_{11} and b_{33} are momentum operators, while b_{13}, b_{31} are winding operators.

Winding background, IIB$_w$. Choosing the NSNS winding modes, however, we find that they are not local with respect to the ground ring generator a_{11}. We conclude that in this case the ground ring is generated by the elements b_{11}, b_{13}, b_{31} and b_{33}, satisfying the relation

$$b_{11}b_{33} - b_{13}b_{31} = 0\,. \tag{2.29}$$

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2.3.2 Type IIA

We project with respect to $S^+(z)$ in the left moving sector and with respect to $\bar{S}^-(\bar{z})$ defined by (2.5) in the right moving sector. The ground ring elements that survive the GSO projection are

$$a_{ij} = \begin{pmatrix} & \mathbf{x\bar{y}} \\ & \end{pmatrix}, \qquad b_{kl} = \begin{pmatrix} \mathbf{u\bar{u}} & & u\bar{v} \\ & & \\ \mathbf{v\bar{u}} & & \mathbf{v\bar{v}} \end{pmatrix}, \tag{2.30}$$

where again we denoted by bold letters the generators of the ground ring.

Momentum background, IIA$_m$. Checking locality with respect to the spectrum found in section 2.2, we find that a_{12} fails to be local, whereas the remaining generators are local. In analogy to the type IIB case above, we find that the ground ring is generated by four elements $b_{11}, b_{13}, b_{31}, b_{33}$ with the relation

$$b_{11}b_{33} - b_{13}b_{31} = 0\,. \tag{2.31}$$

Winding background, IIA$_w$. Here, the ground ring is generated by four elements $a_{12}, b_{31}, b_{11}, b_{33}$ with the relation

$$(a_{12})^2 b_{31} - b_{11}b_{33} = 0\,. \tag{2.32}$$

Ground ring singularity. The (complex) equations (2.28) and (2.32) define a singular Calabi-Yau 3-fold known as the suspended pinch point singularity, whereas the (complex) equations (2.29) and (2.31) define the ubiquitous conifold.

2.3.3 Deformation of the singularity

Conifold singularity. Let us first discuss the deformation of the conifold equation (2.29) and (2.31). A deformation consistent with the quantum numbers, i.e. the x and ϕ momenta and the periodicity sector, of $b_{11}b_{33}$ ($b_{13}b_{31}$) is given by

$$b_{11}b_{33} - b_{13}b_{31} = \mu^2\,, \tag{2.33}$$

where μ couples to the zero x-momentum RR operator in the respective theory. In type IIB$_w$ it is

$$V_{\rm RR}^{\rm IIB} = e^{-\frac{\sigma}{2} - \frac{\bar{\sigma}}{2} + \frac{i}{2}(H+\bar{H}) + \phi}\,, \tag{2.34}$$

while in type IIA$_m$ it reads

$$V_{\rm RR}^{\rm IIA} = e^{-\frac{\sigma}{2} - \frac{\bar{\sigma}}{2} + \frac{i}{2}(H-\bar{H}) + \phi}\,. \tag{2.35}$$

We therefore suggest that the type IIB$_w$ and IIA$_m$ theories deformed by these RR operators are equivalent to the topological B-model on the deformed conifold.

Let us perform some consistency checks of the proposal. Consider first the sphere partition function. In [8] the sphere partition function of the $2d$ type II string has been argued to vanish. The reason is the existence of four fermionic zero modes, compared to the two in the $N = 1$ Liouville system where the sphere partition function is finite. We expect a non-vanishing sphere partition function of the corresponding topological B-model, which implies a lift or projection of the two extra fermionic zero modes. Indeed, the RR deformation we put breaks $N = 2$ supersymmetry on the worldsheet and therefore we expect the additional zero modes to be lifted. Hence, we can expect a non-vanishing sphere partition function also on the non-critical string side of our correspondence.

Next, let us compare the one-loop partition functions of the deformed type IIB$_w$ and IIA$_m$ theories, with the one-loop partition of the topological B-model on the deformed conifold. The one-loop partition function of the B-model is given by $-\frac{1}{12}\log\hat{\mu}$, where $\hat{\mu}$ is the deformation parameter of the conifold. According to (2.33), $\hat{\mu} = \mu^2$, and therefore

$$Z_{\mathrm{B-model}} = -\frac{1}{6}\log\mu\,. \tag{2.36}$$

The calculation of the one-loop partition function in the type II non-critical string, on the other hand, is a rather subtle issue. It was recently proposed [13] that the correct result is given by the regularized sum over the physical spectrum, i.e. the spectrum obtained after imposing the Dirac constraint. However, we would like to propose that in the type IIA/B models at hand the one-loop partition sum is obtained, as usually done, by summing over the spectrum without imposing the Dirac constraint.

Indeed, if we apply this prescription to the spectra of the IIA (IIB) momentum (winding) background we find

$$Z_{\mathrm{IIA_m,IIB_w}} = -\frac{1}{6}\log\mu\,, \tag{2.37}$$

with the $\log\mu$ giving the Liouville volume, as determined by the Liouville momentum one operators (2.34) and (2.35). With this prescription we find agreement between the one-loop partition functions of the topological B-model and the deformed type IIA/B theories. Note, that if instead we summed only over the physical spectra (2.14) and (2.15), we would get for the type IIA/B theories $-\frac{1}{24}\log\mu$. It remains an important open problem to calculate the one-loop partition function in detail to verify the prescription.

Suspended pinch point singularity. Consider next the singularities (2.28) and (2.32), which are of the type

$$x^2y - zw = 0\,. \tag{2.38}$$

The singularity of (2.38) at the origin can be resolved once by changing coordinates to $z' = z/x$. z' is a coordinate on a 2-sphere that we grow at the origin. With this change of variables equation (2.38) becomes

$$xy - z'w = 0 \tag{2.39}$$

which we recognize as the conifold equation.

We can deform the suspended pinch point singularity in two ways. One is by

$$x^2y - zw = \hat{\mu}\,, \tag{2.40}$$

and the second by

$$x^2y - zw = \mu x\,. \tag{2.41}$$

The latter deformation has been discussed in [17]. As we will show later, the deformation (2.41) arises naturally in the framework of the corresponding quiver gauge theories, where μ is related to the dynamical scale Λ and to the glueball superfield S. It is also related to the deformation of the conifold equation (2.39) by the change of variables discussed above. Locally the deformed space looks like T^*S^3, where μ is the scale of S^3. Globally it is different since, in particular, $h^{1,1} = 1$ for the deformed singularity (2.41).

Consider the two types of deformation (2.40) and (2.41) in the type IIB$_m$ and type IIA$_w$ theories. Let us start with the deformation (2.40). In type IIB$_m$ the ground ring relation (2.28) is deformed as

$$(a_{11})^2 b_{33} - b_{13} b_{31} = \mu^2 \,, \tag{2.42}$$

where μ couples to the RR operator (2.34).

Similarly, in type IIA$_w$ the ground ring relation is deformed as

$$(a_{12})^2 b_{31} - b_{11} b_{33} = \mu^2 \,, \tag{2.43}$$

where μ couples to the RR operator (2.35).

We therefore suggest that the type IIB$_m$ and IIA$_w$ theories deformed by these RR operator are equivalent to the topological B-model on the deformed suspended pinch point singularity (2.40).

Again, we can compute the one-loop partition function by summing over the spectra and get

$$Z_{\mathrm{IIA_w,IIB_m}} = -\frac{1}{6} \log \mu \,. \tag{2.44}$$

However, we do not know what is the one-loop partition function of the topological B-model on the deformed singularity (2.40). Following the proposed correspondence we suggest that it is given by $-\frac{1}{12} \log \hat{\mu}$, exactly as in the conifold.

Consider next the deformation (2.41). In order to get in type IIB$_m$ a deformation of the ground ring relation to

$$(a_{11})^2 b_{33} - b_{13} b_{31} = \mu a_{11} \,, \tag{2.45}$$

we need that μ couples to the RR operator with $\frac{1}{2}$ x-momentum and $\frac{3}{2}$ ϕ-momentum. Looking at the spectra (2.13) we see that there is no such operator. However, if we have the coupling $\hat{\mu} \int V^{\mathrm{IIB}}_{\mathrm{NSNS}} + \hat{\mu} \int V^{\mathrm{IIB}}_{\mathrm{RR}}$ where

$$V^{\mathrm{IIB}}_{\mathrm{NSNS}} = e^{-\sigma - \bar{\sigma} + \frac{i}{2}(X_L + X_R) + \frac{\phi}{2}} \,, \qquad V^{\mathrm{IIB}}_{\mathrm{RR}} = e^{-\frac{\sigma}{2} - \frac{\bar{\sigma}}{2} + \frac{i}{2}\left(H + \bar{H}\right) + \phi} \,, \tag{2.46}$$

then the perturbation $\mu \int V^{\mathrm{IIB}}_{\mathrm{NSNS}} \int V^{\mathrm{IIB}}_{\mathrm{RR}}$ with $\mu = \hat{\mu}^2$ satisfies the requirements.

Similarly, in type IIA$_w$ in order to get a deformation of the ground ring relation to

$$(a_{12})^2 b_{31} - b_{11} b_{33} = \mu a_{12} \,, \tag{2.47}$$

we suggest the coupling $\hat{\mu} \int V^{\mathrm{IIA}}_{\mathrm{NSNS}} + \hat{\mu} \int V^{\mathrm{IIA}}_{\mathrm{RR}}$ such that $\mu = \hat{\mu}^2$, and

$$V^{\mathrm{IIA}}_{\mathrm{NSNS}} = e^{-\sigma - \bar{\sigma} + \frac{i}{2}(X_L - X_R) + \frac{\phi}{2}} \,, \qquad V^{\mathrm{IIA}}_{\mathrm{RR}} = e^{-\frac{\sigma}{2} - \frac{\bar{\sigma}}{2} + \frac{i}{2}\left(H - \bar{H}\right) + \phi} \,. \tag{2.48}$$

We therefore propose that the type IIB_m and IIA_w theories deformed by the above operators are equivalent to the topological B-model on the deformed suspended pinch point singularity (2.41).

Again, we can compute the one-loop partition function by summing over the spectra, and get

$$Z_{\mathrm{IIA_w,IIB_m}} = -\frac{1}{12}\log\mu\,, \tag{2.49}$$

where $-\frac{1}{6}$ is the regularized summation over the spectra, and the Liouville volume is $\log\hat{\mu} = \frac{1}{2}\log\mu$. To compare this result to the one-loop partition function of the B-model on the deformed suspended pinch point (2.41), we recall that in the patch $x \neq 0$ the geometry is described by the conifold equation. Hence, we expect the usual conifold result $-\frac{1}{12}\log\mu$. This will be justified more thoroughly in the last section where we also analyze the corresponding quiver gauge theory.

Note, that we can perturb by a single RR operator, as well. However, since the required ϕ momentum is bigger than $\frac{Q}{2}$, this operator is in the "non-local" regime [11]. In type IIB_m such an operator is

$$V^{\mathrm{IIB}}_{\mathrm{RR\ non-local}} = e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H+\bar{H}\right)+\frac{i}{2}(X_L+X_R)+\frac{3}{2}\phi}\,, \tag{2.50}$$

while in type IIA_w it reads

$$V^{\mathrm{IIA}}_{\mathrm{RR\ non-local}} = e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+\frac{i}{2}(X_L-X_R)+\frac{3}{2}\phi}\,. \tag{2.51}$$

One may try to understand such a deformation, with the operator inserted in the worldsheet path integral being $\int a_{11}^{-1} V^{\mathrm{IIB}}_{\mathrm{RR\ non-local}} \sim e^{2\phi}+...$ in type IIB, and similarly $a_{12}^{-1}V^{\mathrm{IIA}}_{\mathrm{RR\ non-local}}$ in type IIA.

It is amusing to check the one-loop partition functions. Indeed, we will get the required result if we take the Liouville volume to be $\frac{1}{2}\log\mu$, which may be plausible since the perturbations $a_{11}^{-1}V^{\mathrm{IIB}}_{\mathrm{RR\ non-local}}$ and $a_{12}^{-1}V^{\mathrm{IIA}}_{\mathrm{RR\ non-local}}$ carry Liouville momentum two.

2.4 $\mathcal{Q}$ Projection

The space-time supercharge $\mathcal{Q}$, obtained by the closed contour integral of the supercurrent S

$$\mathcal{Q} = \oint \frac{dz}{2\pi i} S(z)\,, \tag{2.52}$$

is nilpotent $\mathcal{Q}^2 = 0$. Hence, $\mathcal{Q}$ can be considered as a BRST-like charge. One can consider a projected theory obtained by restricting to ker $\mathcal{Q}$, with physical operators satisfying

$$\mathcal{Q}|\mathrm{phys}\rangle = 0\,. \tag{2.53}$$

The projected theories have been suggested as possible topological theories [10, 12]. Note that these theories are also candidates for a topological B-model correspondence, since the ground ring is invariant under the $\mathcal{Q}$-projection [16]. In this section we will consider the

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spectra of these theories. However, the results cast doubts on the consistency of these theories. Still, these results must be taken with a grain of salt. More conclusive statements require a fully-fledged derivation of the partition function for these theories.

We will first discuss the differences between the projections by $\mathcal{Q}^+$ and $\mathcal{Q}^-$ in the various chiral sectors. Then we will combine the $\mathcal{Q}$-projected sectors to find the spectra of the projected type IIA/B theories.

In the NS sector the tachyon vertex operators T_k are of the form (2.7). Projecting onto the operators that satisfy $\mathcal{Q}^+T_k = 0$, with $\mathcal{Q}^+$ given by the holomorphic integral of the current (2.3) implies

$$\mathcal{Q}^+T_k = 0 \rightarrow \quad k \in \mathbb{Z} + \frac{1}{2} > 0\,. \tag{2.54}$$

On the other hand, using $\mathcal{Q}^-$ for the projection gives the opposite condition

$$\mathcal{Q}^-T_k = 0 \rightarrow \quad k \in \mathbb{Z} + \frac{1}{2} < 0\,. \tag{2.55}$$

In the Ramond sector we have the vertex operators V_k of the form (2.10), with $\epsilon = \pm 1$. The states of the $\epsilon = 1$ sector, which are invariant under $\mathcal{Q}^+$, satisfy

$$\mathcal{Q}^+V_k^{\epsilon=1} = 0 \rightarrow \quad k \in \mathbb{Z} \geq 0\,, \tag{2.56}$$

whereas for the $\epsilon = -1$ sector we find

$$\mathcal{Q}^+V_k^{\epsilon=-1} = 0 \rightarrow \quad k \in \mathbb{Z} + \frac{1}{2} > 0\,. \tag{2.57}$$

The operator $\mathcal{Q}^-$ again acts in the opposite way and the $\mathcal{Q}^-$ invariant operators are given by

$$\begin{aligned} \mathcal{Q}^-V_k^{\epsilon=1} &= 0 \rightarrow k \in \mathbb{Z} + \frac{1}{2} < 0\,, \\ \mathcal{Q}^-V_k^{\epsilon=-1} &= 0 \rightarrow k \in Z \leq 0\,. \end{aligned} \tag{2.58}$$

Let us now combine the left and right movers to get the physical closed string states on the $\mathcal{Q}$-projected theories. In the following we will present the results after imposing the Dirac constraint. In the appendix we list the spectra before imposing it. First we consider the theories obtained by using just the holomorphic $\mathcal{Q}$ operator. We obtain the following spectra consistent with the Dirac constraint

- $\mathrm{IIB}_m^{\mathcal{Q}}$:

$$\begin{aligned} NSNS \quad & e^{-\sigma-\bar\sigma+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, \qquad n \in \mathbb{Z} \geq 0\,, \\ RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L+X_R)}\,, \qquad m \in \mathbb{Z} \geq 0\,. \end{aligned} \tag{2.59}$$

- $\mathrm{IIB}_w^{\mathcal{Q}}$:

$$\begin{aligned} NSNS \quad & e^{-\sigma-\bar\sigma+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, \qquad n \in \mathbb{Z} \geq 0\,, \\ RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar\sigma}{2}+\frac{i}{2}\left(H+\bar H\right)+im(X_L+X_R)}\,, \qquad m \in \mathbb{Z} \geq 0\,. \end{aligned} \tag{2.60}$$

- $\mathrm{IIA}_m^{\mathcal{Q}}$:

$$
\begin{aligned}
NSNS \quad & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, && n\in\mathbb{Z}\geq 0\,,\\
RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, && m\in\mathbb{Z}\geq 0\,.
\end{aligned}
\tag{2.61}
$$

- $\mathrm{IIA}_w^{\mathcal{Q}}$:

$$
\begin{aligned}
NSNS \quad & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, && n\in\mathbb{Z}\geq 0\,,\\
RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, && m\in\mathbb{Z}\geq 0\,.
\end{aligned}
\tag{2.62}
$$

We can also define theories by acting with both the holomorphic $\mathcal{Q}$ and the antiholomorphic $\bar{\mathcal{Q}}$. For these symmetric projections we find

- $\mathrm{IIB}_m^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$
\begin{aligned}
NSNS \quad & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, && n\in\mathbb{Z}\geq 0\,,\\
RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L+X_R)}\,, && m\in\mathbb{Z}\geq 0\,.
\end{aligned}
\tag{2.63}
$$

- $\mathrm{IIB}_w^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$
RR \quad e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L+X_R)}\,, \qquad m\in\mathbb{Z}\geq 0\,.
\tag{2.64}
$$

- $\mathrm{IIA}_m^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$
RR \quad e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, \qquad m\in\mathbb{Z}\geq 0\,.
\tag{2.65}
$$

- $\mathrm{IIA}_w^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$
\begin{aligned}
NSNS \quad & e^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, && n\in\mathbb{Z}\geq 0\,,\\
RR \quad & e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, && m\in\mathbb{Z}\geq 0\,.
\end{aligned}
\tag{2.66}
$$

Note that none of the above theories features space-time fermions since we are considering the $\mathcal{Q}$ cohomology and the NSR states are $\mathcal{Q}$-exact.

The partition function. Let us turn now to the torus partition function. Due to the absence of transverse oscillations we assume that the partition functions in the $\mathcal{Q}$-projected theories reduce to sums over discrete momenta. For the theories to be consistent this sum has to be modular invariant. However, as we show in the appendix, the sum fails to do so, which suggests that the theories obtained by the $\mathcal{Q}$-projection may fail to be consistent.

3. Quiver gauge theories

We consider the quiver gauge theory obtained by D-branes wrapping in the suspended pinch point geometry (see [18–20]). Integrating out the D-brane open string degrees of freedom results in a deformed geometry interpreted as the backreaction of the D-branes on the original geometry. This back reaction, in the topological strings framework, affects the closed strings by changing the periods of the holomorphic 3-form. The Calabi-Yau space undergoes a transition, where holomorphic cycles shrink and 3-cycles are opened up. In the language of four-dimensional $\mathcal{N}=1$ supersymmetric gauge theory, the D-branes wrapping the resolved geometry provide the UV description, while the IR physics is described by the deformed geometry after the transition.

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3.1 Toric geometry

The suspended pinch point singularity is described by the toric data

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{array}{c} \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 \end{pmatrix} \end{array} . \tag{3.1}$$

These charges represent a $\mathrm{U}(1)^2$ action on the coordinates $z_i \in \mathbb{C}^5$. The singular space is the symplectic quotient of $\mathbb{C}^5$ by $U(1)^2$ defined by the (D-term) equations

$$\begin{aligned} |z_1|^2 + |z_3|^2 - |z_4|^2 - |z_5|^2 &= 0\,, \\ |z_2|^2 - |z_3|^2 + |z_4|^2 - |z_5|^2 &= 0\,, \end{aligned} \tag{3.2}$$

modulo the $\mathrm{U}(1)^2$ action. The resolved space is obtained by introducing two (FI) parameters t_1, t_2 , such that

$$\begin{aligned} |z_1|^2 + |z_3|^2 - |z_4|^2 - |z_5|^2 &= t_1\,, \\ |z_2|^2 - |z_3|^2 + |z_4|^2 - |z_5|^2 &= t_2\,. \end{aligned} \tag{3.3}$$

In order to see that indeed the space we are considering is the 3-fold described by the ground ring of the type II string, we will describe the toric singularity in terms of a collection of polynomial equations, written in terms of $U(1)^2$-invariant monomials of the coordinates z_i. The monomials are generated by the four elements

$$a_{22} = z_1 z_2 z_5\,, \quad b_{11} = z_3 z_4\,, \quad b_{13} = z_1^2 z_4 z_5\,, \quad b_{31} = z_2^2 z_3 z_5\,, \tag{3.4}$$

and satisfy (2.28). In the geometric transition, an S^3 cycle opens up, and the relation (2.28) is deformed to (2.41).

3.2 Quiver gauge theories

The UV description of the quiver gauge theories is found by D-branes in the resolved geometry. Consider a system of D3-branes placed at the singularity (3.2), and fractional D3-branes obtained by wrapping D5-branes on the two $\mathbb{P}^1$'s. The quiver gauge theory reads

	$\mathrm{SU}(N_1)$	$\mathrm{SU}(N_2)$	$\mathrm{SU}(N_3)$
X	$\square$		$\bar{\square}$
$\tilde{X}$	$\bar{\square}$		$\square$
Y		$\square$	$\bar{\square}$
$\tilde{Y}$		$\bar{\square}$	$\square$
Z	$\square$	$\bar{\square}$	
$\tilde{Z}$	$\bar{\square}$	$\square$	
Φ			Adj.

(3.5)

We will consider the setup where $N_1 = N_3 \neq N_2$, obtained by placing $N_1 = N_3$ D3-branes at the singularity and wrapping $N_2 - N_1$ D5-branes on one $\mathbb{P}^1$.

The classical superpotential is

$$W_{\text{tree}} = \mathrm{tr}\left[\Phi(\tilde{Y}Y - \tilde{X}X) + (Z\tilde{Z}X\tilde{X} - \tilde{Z}ZY\tilde{Y})\right]. \tag{3.6}$$

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3.2.1 Ring structure

The classical ring relations are obtained using the holomorphic gauge invariants

$$x = X\tilde{X} = Y\tilde{Y}\,, \qquad y = Z\tilde{Z}\,, \qquad z = X\tilde{Y}\tilde{Z}\,, \qquad w = ZY\tilde{X}\,, \tag{3.7}$$

which satisfy $x^2y - zw = 0$. In order to see the quantum deformation one considers $N_2 = N_c$ and replaces the two other groups by $U(1)'s$ (D3-brane probes). In terms of the meson

$$M_{ij} = \begin{pmatrix} Z\tilde{Z} & ZY \\ \tilde{Y}\tilde{Z} & \tilde{Y}Y \end{pmatrix}, \tag{3.8}$$

the nonperturbative superpotential is given by

$$W_{np} = (N_c - 2)\left(\frac{\Lambda^{3N_c-2}}{det\,M}\right)^{\frac{1}{N_c-2}} + \mathrm{tr}\left[\Phi(M_{22} - \tilde{X}X) + (M_{11}X\tilde{X} - M_{12}M_{21})\right]\,, \tag{3.9}$$

which deforms the ring relation to

$$x^2y - zw = \mu\, x, \quad \mu = \left(\Lambda^{3N_c-2}\right)^{\frac{1}{N_c-1}}\,. \tag{3.10}$$

3.2.2 Holomorphic F-terms and B-model free energy

Let us consider now the IR dynamics of the quiver gauge theory. After running the duality cascade for the quiver gauge theory, one ends up in the IR with a confining pure SYM gauge theory. In the suspended pinch point deformed geometry, we identify the size of S^3 with the glueball superfield S. The holomorphic F-terms of this theory $\mathcal{F}_{\mathrm{SYM}}(S)$ as a function of S is

$$\mathcal{F}_{\mathrm{SYM}}(S) = F_{c=1}(R_{\mathrm{self-dual}})\,, \tag{3.11}$$

where $S \sim \mu$, and

$$F_{c=1}(R_{\mathrm{self-dual}}) = \frac{1}{2}\mu^2\log\mu - \frac{1}{12}\log\mu + \frac{1}{240}\mu^{-2} + \sum_{g>2} a_g\mu^{2-2g}\,, \tag{3.12}$$

where $a_g = \frac{B_{2g}}{2g(2g-2)}$.

We therefore propose that this is also the perturbative free energy of the topological B-model on the deformed suspended pinch point geometry (2.41). This seems very plausible, since locally the deformed suspended pinch point singularity looks like the deformed conifold T^*S^3, and the perturbative free energy of the topological B-model on the deformed conifold is given by (3.12). However, since the deformed conifold and the deformed suspended pinch point geometries are globally different, one should expect that the free energy of the topological B-models on these geometries will differ non-perturbatively,

Returning now to the perturbed type II strings, as discussed in section 2.3.3, we propose that in all cases described there, the perturbative free energy is also given by (3.12), with the appropriate map of the deformation parameters as discussed there. We argued for the consistency of this proposal at tree level and one-loop. Going beyond that will probably require a matrix model description of the system, which is currently lacking.[1] It would also be interesting to construct the target space geometry of the perturbed two-dimensional strings.

[1] We refer the reader to [21 − 23] and the references therein for related results in this direction.

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A. $\mathcal{Q}$ projection

In this appendix we list the spectra of the type II theories, which are consistent with the holomorphic $\mathcal{Q}$ projection. We do not impose the Dirac condition here.

- $\mathrm{IIB}^{\mathcal{Q}}_{m}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L\pm X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H+\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,, \end{aligned}\tag{A.1}$$

- $\mathrm{IIB}^{\mathcal{Q}}_{w}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L\pm X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H+\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,, \end{aligned}\tag{A.2}$$

- $\mathrm{IIA}^{\mathcal{Q}}_{m}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L\pm X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,, \end{aligned}\tag{A.3}$$

- $\mathrm{IIA}^{\mathcal{Q}}_{w}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L\pm X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,. \end{aligned}\tag{A.4}$$

B. $\mathcal{Q}$, $\bar{\mathcal{Q}}$ projection

In this appendix we have compiled the list of spectra consistent with the combined holomorphic/antiholomorphic $\mathcal{Q}$, $\bar{\mathcal{Q}}$-projection. Again, the Dirac condition is not imposed.

- $\mathrm{IIB}^{\mathcal{Q},\bar{\mathcal{Q}}}_{m}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L+X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H+\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L+X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,, \end{aligned}\tag{B.1}$$

- $\mathrm{IIB}_w^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H+\bar{H}\right)+im(X_L+X_R)}\,, \qquad m\in\mathbb{Z}\geq 0\,, \tag{B.2}$$

- $\mathrm{IIA}_m^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, \qquad m\in\mathbb{Z}\geq 0\,, \tag{B.3}$$

- $\mathrm{IIA}_w^{\mathcal{Q},\bar{\mathcal{Q}}}$:

$$\begin{aligned} NSNSe^{-\sigma-\bar{\sigma}+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,,\\ RRe^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}+\frac{i}{2}\left(H-\bar{H}\right)+im(X_L-X_R)}\,, &\qquad m\in\mathbb{Z}\geq 0\,,\\ e^{-\frac{\sigma}{2}-\frac{\bar{\sigma}}{2}-\frac{i}{2}\left(H-\bar{H}\right)+i\left(n+\frac{1}{2}\right)(X_L-X_R)}\,, &\qquad n\in\mathbb{Z}\geq 0\,. \end{aligned} \tag{B.4}$$

C. $\mathcal{Q}$-Projection and modular properties

In the following we will only discuss the momentum-type theories but similar arguments apply to the winding-type theories. Consider first the type IIB theory with the $\mathcal{Q}$-projection acting only on the left-moving side. The total partition sum splits into sums over the possible spin structures. Explicitly, these sums are given by

$$\begin{aligned} Z_{\mathrm{NSNS}} &= \sum_{s\geq 0, m\in\mathbb{Z}} q^{\frac{1}{2}\left(s+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(s-2m+\frac{1}{2}\right)^2}\,,\\ Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=1} &= \sum_{s\geq 0, m\in\mathbb{Z}} q^{\frac{1}{2}s^2}\bar{q}^{\frac{1}{2}(s-2m)^2}\,,\\ Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=-1} &= \sum_{s\geq 0, m\in\mathbb{Z}} q^{\frac{1}{2}\left(s+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(s-2m+\frac{1}{2}\right)^2}\,, \end{aligned} \tag{C.1}$$

Since the spectrum is given by the $\mathcal{Q}$ cohomology, the space-time fermions are projected out. As usual, they are $\mathcal{Q}$-exact states.

To exhibit the properties of the above sums under modular transformations we will rewrite them in terms of theta functions $\theta_i(\tau)$. This leads to the following expressions

$$\begin{aligned} Z_{\mathrm{NSNS}} &= Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=-1} == \frac{1}{4}\left|\theta_2\right|^2\,,\\ Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=1} &= \frac{1}{4}\left(\bar{\theta}_3+\bar{\theta}_4\right)+\frac{1}{4}\left(\left|\theta_3\right|^2+\left|\theta_4\right|^2\right)\,. \end{aligned} \tag{C.2}$$

From these expressions it is obvious that the sum over the spin structures is not modular invariant. The most important problem is the sum over the RR sector with $\epsilon_L=\epsilon_R=1$ which includes a zero momentum mode. The sum of this mode over the right sector gives rise to the problematic $\bar{\theta}_3+\bar{\theta}_4$ contribution.

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$\mathcal{Q}$-projection on both sides. If we follow the discussion of section 2.4 and apply the $\mathcal{Q}$-projection on both sides in the type IIB theory, we get the following sums

$$\begin{aligned}
Z_{\mathrm{NSNS}} &= \sum_{m,n\geq 0}\left[q^{\frac{1}{2}\left(2m+1+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(2n+1+\frac{1}{2}\right)^2}+q^{\frac{1}{2}\left(2m+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(2n+\frac{1}{2}\right)^2}\right]\\
&= \frac{1}{4}\left|\theta_2\right|^2-\sum_{m,n\geq 0}\left[q^{\frac{1}{2}\left(2m+1+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(2n+\frac{1}{2}\right)^2}+q^{\frac{1}{2}\left(2m+\frac{1}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(2n+1+\frac{1}{2}\right)^2}\right]\\
&= Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=-1},\\
Z_{\mathrm{RR}}^{\epsilon_L=\epsilon_R=1} &= \sum_{m,n\geq 0}\left[q^{\frac{1}{2}(2m)^2}\bar{q}^{\frac{1}{2}(2n)^2}+q^{\frac{1}{2}(2m+1)^2}\bar{q}^{\frac{1}{2}(2n+1)^2}\right]\\
&= \frac{1}{4}+\frac{1}{8}\left[\theta_3+\bar{\theta}_3+\theta_4+\bar{\theta}_4+\left|\theta_3\right|^2+\left|\theta_4\right|^2\right]. \qquad \text{(C.3)}
\end{aligned}$$

We can argue that even without expressing the remaining sums in terms of theta-functions, the sum over the spin structures fails to be modular invariant. This is because, in the above expression, the purely left (right) moving sums in the RR sector are not modular covariant. Thus, for the total sum to be modular covariant compensating purely left (right) moving sums are required. However, since the remaining sums are over half-integer momenta there are no such purely left (right) moving sums. Therefore, the total sum is not modular covariant.

A similar analysis shows that the partition functions of the $\mathcal{Q}$-projected type IIA theories are not modular invariant. Indeed, we can write the sum of the $\mathcal{Q}$-projected type IIA/B partition functions in a fairly nice form as

$$Z_{\mathrm{IIA}}+Z_{\mathrm{IIB}}=\frac{1}{2}+\frac{1}{2}\left|\theta_2\right|^2\frac{1}{4}\left(\left|\theta_3\right|^2+\left|\theta_4\right|^2\right)+\frac{1}{4}\left(\theta_3+\bar{\theta}_3+\theta_4+\bar{\theta}_4\right). \qquad \text{(C.4)}$$

Note that this expression is not modular covariant. Since we expect either both or neither of them to be modular covariant, the above expression provides another hint that these theories may lack a modular invariant spectrum and are potentially inconsistent.

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